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Colonel Richard's Game
Part I: Elementary Version

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Lincoln Laboratory

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FOR THE COMMANDER

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COLONEL RICHARD'S GAME
PART I: ELEMENTARY VERSION

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Division 3

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ABSTRACT

Colonel Richard's Game, a two-player, single-stage, zero-sum abstract game, is defined. It is based on a military offense-defense situation and is a variant of the classical Colonel Blotto's Game.

Colonel Richard's Game is described, an elementary version is solved, and extension to more complex versions discussed.

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1. INTRODUCTION

A two-person, single-stage game, 'Colonel Richard's Game', is defined and examined. The game is a variant of one known in the literature as 'Colonel Blotto's Game'; the latter game is discussed first and then the variant.

A knowledge of elementary Game Theory is assumed.

2. COLONEL BLOTTO'S GAME

This game, well known to game-theoreticians, is played between two antagonists, Blue and Red. Colonel Blotto, leader of the Blue forces, contends with his opposite number, Colonel Kije of the Red forces, for control of a number of passes through the mountain range that separates the two armies.

Blotto can dispose of N units of military force, while Kije can dispose of M units. Each commander allocates his units among the mountain passes: so many to Pass 1, so many to Pass 2, etc. On the day of confrontation, control of a pass is won by whichever party has more units of force at that position. If the Blue and Red forces at a pass are equinumerous, neither side controls the pass. Blue gains one point for each pass it controls and loses one point for each pass that Red controls.

Blotto disposes his forces to maximize his gain and Kije his forces to minimize Blue's gain. In general, optimal play requires mixed, rather than pure, strategies for each side. The salient aspects of a Blotto game, for our purposes are:

(1) Each commander has finite, discrete resources that are partitioned among a limited number of stations ('passes').

(2) The gain resulting from the confrontation at a station depends only on the Blue and Red forces there. That is, conditions at one station do not influence the outcome at another station.

3. COLONEL RICHARD'S GAME

A military engagement problem propounded by Dr. Richard S. Ruffine, USDDRE, leads naturally to a variant of Colonel Blotto's Game, and has been named 'Colonel Richard's Game' by the writer of this report.

Colonel Richard's Game shares with Colonel Blotto's Game the first aspect mentioned above (namely, that of finite, discrete resources), but differs in the second aspect: that is, conditions at one station can influence the results at another.

In its full ramifications, Colonel Richard's Game is difficult to solve; the present report deals with an elementary version of the game. Some results are given for the elementary game, both for their intrinsic interest and to introduce the new game to the reader, in anticipation of future results on more complex versions of the game.

3.1. RULES OF COLONEL RICHARD'S GAME

Two forces, Blue and Red, are in contention: Red uses interceptor weapons to defend a base that Blue will attack with missiles. The base, sited on an island, is protected by a radar that can detect Blue missiles and guide interceptors to destroy them. Blue has a radar-attack boat (RAB) from which missiles can be fired at the radar in an attempt to render it inoperative, and Red can try to prevent this by allocating interceptors to defend the sensor. The remaining Blue missiles subsequently are fired from a base-attack boat (BAB) against the Red base itself, to be countered by the remaining Red interceptors.

The two phases of the battle are governed by the following rules:

Phase 1: Some (or none) of the Blue missiles are launched against the radar, which is defended by some (or none) of the Red interceptors. If the number of attacking missiles exceeds the number of defending interceptors, the radar is destroyed, otherwise it is unharmed.

Phase 2: The remaining Blue missiles are then launched against the Red base, which is defended by the remaining Red interceptors. If the radar was rendered *hors de combat* in Phase 1, all missiles reach their target; if the radar is still operational, each Red interceptor neutralizes one Blue missile, and only the excess missiles, if any, reach their target.

The following rules apply to the battle as a whole:

Allocation: Blue may partition his missiles between his RAB and BAB in any way, but the missiles on the RAB cannot be used against the base, nor those on the BAB against the radar. Similarly, after Red has assigned certain interceptors to protect the radar, he cannot use them to protect the base, nor can the base interceptors protect the radar.

Scoring: Blue scores one point for each missile that reaches its target at the Red base. Destroying the radar does not, *per se*, gain points for Blue.

Foreknowledge: Each side knows the opponent's stockpile of weapons, but not the allocation of weapons between Phases 1 and 2.

While, in the general version of Colonel Richard's Game, Blue may have several RABs and several BABs, and Red may have several radars, in this report we consider only an elementary version of the game, namely, one in which Blue has one RAB and one BAB, and Red one radar. In the main body of the report, we treat the case in which Blue and Red have the same number of weapons; finally, we extend our results to the case where they have different numbers of weapons.

3.2. PAYOFF MATRIX OF THE ELEMENTARY GAME, EQUAL STOCKPILES

The elementary game, if Blue and Red have N weapons each, has a payoff or game matrix, G(N), of the form:

$$G(N) = \begin{bmatrix} 0 & 1 & 2 & 3 & \dots & N \\ N-1 & 0 & 1 & 2 & \dots & N-1 \\ N-2 & N-2 & 0 & 1 & \dots & N-2 \\ \dots & & & & & \\ 1 & 1 & 1 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

The entry in the i^{th} row and j^{th} column of G(N), g_{ij}, is the gain to Blue (= the loss to Red) if Blue fires i missiles at the radar and Red defends that sensor with j interceptors ($0 \leqslant i, j \leqslant N$)*. Thus:

(a) The entries on the main diagonal, $g_{ii} = 0$, correspond to the outcome in which the i missiles attacking the radar in Phase 1 are successfully countered by the i interceptors defending it, and the N – i missiles attacking the base in Phase 2 are neutralized (with the assistance of the radar) by the N – i interceptors defending the base. Blue's gain is 0, because no missiles reach the base.

(b) The entries above the main diagonal:

$$g_{ij} = j - i; \qquad j > i$$

correspond to the outcome in which Red allocates more interceptors to defend the radar than Blue sends missiles to attack it. The radar survives to guide the base interceptors, but there are fewer of these than there are attacking missiles and the excess missiles score. In the extreme case, j = N, in which Red uses every interceptor to defend the radar, every missile that Blue fires against the base scores.

(c) The entries below the main diagonal:

$$g_{ij} = N - i; \qquad j < i$$

correspond to the outcome in which the radar is destroyed because too few interceptors are allocated to defend it. Hence, all the missiles launched against the base score.

* It will be convenient, throughout this report, to enumerate the rows of payoff matrices as starting from the 0^{th} row (not the 1^{st}), and similarly with the columns. The i^{th} row, then, relates to the use of i missiles against the radar, and the j^{th} column to the use of j interceptors to defend the radar. A similar convention applies to vectors, the initial component of which is the 0^{th}.

It is clear, both from the description of the game and from the form of G*, that if Red knew Blue's choice of radar-attack missiles (i.e., the value of i), he would nullify Blue's score by choosing $j = i$; that is, Red would choose as many interceptors to defend the radar as there are attacking missiles. Conversely, if Blue knew Red's choice of radar-defense interceptors (the value of j), he would attack the radar with just the number of missiles — either 0 or $j + 1$, depending on the value of j — to gain the maximum possible score. (Note that Blue would not necessarily choose to destroy the radar.) However, under our assumptions, neither side knows the other's choice and the outcome is problematical.

As we shall see, for the general value of N, there is no saddle point to the game and each party should use a mixed strategy. That is, Blue should choose the value of i from some probability distribution, depending on the value of N; similarly, Red should choose the value of j according to some other stochastic law. If both parties play properly, the resulting value of the game, v, is optimal in the usual game-theoretic minimax sense. That is, v is as large as Blue could reasonably expect in repeated play against an intelligent Red player, and is at the same time as small as Red can reasonably expect to keep it against an intelligent Blue player. By the nature of the game, of course, v is a non-negative number, $0 \leqslant v \leqslant N$.

The solution of the game takes the form of two $(N + 1)$-component *strategy vectors*, B and R. The i^{th} component of B, b_i, is the probability that Blue allocates i missiles to his RAB to attack the radar; the j^{th} component of R, r_j, is the probability that Red allocates j interceptors to defend the radar. Here, $0 \leqslant i, j \leqslant N$, each component of B and R must lie in the interval 0 to 1, and the sum of the components of each vector must be unity.

We remark that not all of a player's available strategies need be represented by positive probabilities in his strategy vector. Some strategies (that is, choices of i or j, as the case may be) might be so poor that they should never be played. Such strategies are represented by '0's in the strategy vector and are known as *inactive* strategies; strategies that are played with some nonzero probability are known as *active* strategies.

3.3. SOLUTION OF THE GAME

Let J be an $(N + 1)$-component row vector consisting of '1's. Then if the game matrix is such that all strategies are active, we can express the value of the game as:

$$v = 1/(JG^{-1}J^t) \tag{1.1}$$

where 't' indicates 'transpose'. Then Blue's strategy vector is:

$$B = vJG^{-1} \tag{1.2}$$

while Red's strategy vector is:

$$R = vJ(G^{-1})^t \quad . \tag{1.3}$$

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* For typographical convenience, we often will write G in place of G(N), where no confusion can result.

(see Reference 1, Chapter 2; Reference 2, Chapter 2; or Reference 3, Chapter 3). Equations (1) can be modified to account for the case of G being singular, but all the matrices we will wish to invert will, in fact, be regular.

We can also determine the mean number of missiles that Blue will launch from the RAB at the radar; it is:

$$E_b = BK^t \tag{2.1}$$

where K is an (N + 1)-component row vector whose i^{th} entry is $k_i = i$.

Similarly, the mean number of interceptors that Red will assign to protect the radar is:

$$E_r = RK^t \tag{2.2}$$

3.4. RESULTS FOR SMALL N

We shall examine the game in detail for some small values of N.

3.4.1. Case, N = 1

This smallest game has the payoff matrix:

$$G(1) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Blue scores only in the circumstance that he sends his single missile to the base while Red uses his single missile to defend the radar. Uniquely, this game has a saddle point, since Red will choose j = 0, and the value of the game is v = 0, regardless of Blue's action.

3.4.2. Case, N = 2

The payoff matrix is:

$$G(2) = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The third column is dominated by each of the other columns and the third row is dominated by each of the other rows. These conditions imply, respectively, that strategies j = 2 and i = 2 are inactive. Discarding these strategies, we have the *reduced matrix*, continuing only active strategies:

$$G' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

whence, from Equations (1) and (2):

$$B = [1 \quad 1 \quad 0]/2; \qquad R = [1 \quad 1 \quad 0]/2$$

and: $$v = 1/2; \qquad E_b = E_r = 1/2 \quad .$$

Note that, in forming B and R, we insert '0's in the locations of the inactive strategies that are not represented in the reduced matrix.

3.4.3. Case, N = 3

The payoff matrix is:

$$G(3) = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Column 3 and Row 3 are dominated: the matrix reduces to:

$$G' = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

We find: $$v = 5/6; \qquad E_r = 5/6; \qquad E_b = 7/6$$

$$B = [2 \quad 1 \quad 3 \quad 0]/6$$

$$R = [2 \quad 3 \quad 1 \quad 0]/6$$

As for the case, N = 2, we have $E_r = v$; that this is true for all G(N) is shown in Appendix A.

3.4.4. Case, N = 4

The payoff matrix is:

$$G(4) = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 3 & 0 & 1 & 2 & 3 \\ 2 & 2 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Strategies i, j = 3, 4 can be deleted by arguments of dominance, leading to the solution:

$$v = 7/6; \qquad E_b = 5/6$$

$$B = [6 \quad 2 \quad 4 \quad 0 \quad 0]/12 = [0.50 \quad 0.17 \quad 0.33 \quad 0 \quad 0]$$

$$R = [3 \quad 4 \quad 5 \quad 0 \quad 0]/12 = [0.25 \quad 0.33 \quad 0.42 \quad 0 \quad 0]$$

The players, it will be noted, will never deploy three or four weapons against or in defense of the radar. Had the inactive status of strategies i, j = 3, 4 not been recognized — that is, had either or both of these strategies been left in the game matrix — then B or R would have been found to contain a negative component: a sure sign that the reduced matrix was not in its final form.

In the game G(N), dominated strategies occur in pairs (for the same values of i and j, that is) and are found among the highest-order row-column pairs. These circumstances make it easy to recognize and delete the inactive strategies.

3.5. HIGHER VALUES OF N

We have solved the elementary version of Colonel Richard's Game for values of N up to 20. The results are shown graphically in Figure 1, and the strategy vectors are listed in Appendix B.

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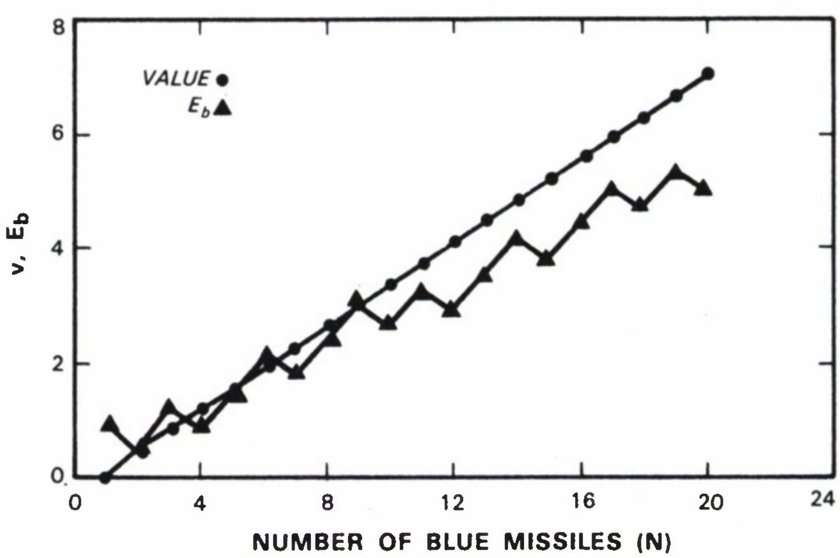

Figure 1. Value of game and number of radar missiles for game G(N).

The value is approximately linear with N, with a slope of 0.366, i.e., each unit increase in N brings Blue an additional return of 0.366 — the more missiles there are in the game, the greater Blue's score. Considering the solutions for $2 \leqslant N \leqslant 20$, the correlation coefficient of v against N is in excess of 0.999; the line of regression of v upon N is given by:

$$v = -0.273 + 0.3655 \times N \quad .$$

Since $E_r = v$, we have just described the behavior of E_r.

The graph of E_b is erratic: it has a general slope of roughly 0.27 — less than that of v — and the local behavior of E_b is nonmonotonic; the staircase behavior is associated with Ac, the number of active Blue (or Red) strategies. An examination of the results in Appendix B discloses that a unit increase in N is, more often than not, accompanied by a unit increase in Ac — that is, the number of *inactive* strategies grows more slowly than N. E_b increases with N except when the number of inactive strategies also increases, and it is precisely at those values that E_b decreases. The behavior of E_b for large values of N is not known.

It is interesting to note the patterns in the structures of the two strategy vectors displayed in Appendix B; they suggest the following optimum strategies for the two contestants:

> Blue should direct all his missiles against Red's base about 40% of the time, and should attack the radar to some degree the remaining 60% of the time. In the latter case, the number of missiles sent against the radar varies from 1 to (Ac – 1), as chosen from a gently increasing ramp of probabilities.
>
> Red should allocate from 0 to (Ac – 1) missiles to defend the radar; the probabilities for these actions form a ramp of linearly increasing values, commonly with a slight dip for the probability of assigning (Ac – 1) missiles.

The consequence of such tactics is that Blue will realize an optimal score of approximately $v = 0.366 \times N$.

3.6. UNEQUAL STOCKPILES

So far, we have considered the case in which Blue and Red have the same number of weapons. We now extend these results to the case of unequal stockpiles. We designate the game in which Blue has N weapons and Red has M by the symbol, G(N, M); this symbol also identifies the payoff matrix of the game. The equinumerous game treated previously is, G(N, N) = G(N).

3.6.1. Blue Has More Weapons than Red (N > M)

The payoff matrix for G(N, M) is of size $(N + 1) \times (M + 1)$, but the additional (N – M) rows are dominated and can be eliminated. The resulting $(M + 1) \times (M + 1)$ reduced matrix is identical to the matrix, G(M), for the equinumerous case, except that each entry has been increased by (N – M). It is easy to show (e.g., exercise in Reference 1, Chapter 1), that the

strategy vectors for G(N, M) are the same as those of G(M) and that the value of the new game is (N - M) more than that of the equinumerous game. This argument disposes of the case $N > M$.

3.6.2. Blue Has Fewer Weapons than Red ($N < M$)

This situation is more complicated, because the form of the payoff matrix changes radically, and there is no simple relation between the new game and the equinumerous game. For example, if Blue has three missiles and Red has four interceptors, the matrix is:

$$G(3, 4) = \begin{bmatrix} 0 & 0 & 1 & 2 & 3 \\ 2 & 0 & 0 & 1 & 2 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad .$$

From considerations of dominance, strategies i = 1, 3 and j = 0, 3, 4, can be eliminated, giving the reduced matrix:

$$G' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad ,$$

which we have already encountered in the game, G(2), but this is a coincidence. The strategy vectors are:

$$B = [1 \quad 0 \quad 1 \quad 0]/2$$

$$R = [0 \quad 1 \quad 1 \quad 0 \quad 0]/2$$

and: $$v = 1/2; \qquad E_b = 1; \qquad E_r = 3/2 \quad ,$$

a solution we have not previously seen. We observe that E_r no longer equals v: indeed, it can be shown, using arguments similar to those in Appendix A, that for the game G(N, M), $E_r = v + (M - N)$, whatever the relative values of N and M, so that we need not calculate E_r independently.

We remark that the structure of the strategy vectors for the game G(3, 4) is characteristic of the solutions for any game, $N < M$, in the following respects:

R consists of an initial block of (M – N) '0's, followed by a block of nonzero coefficients, and terminated by a block of '0's. Note that Red never fails to protect the radar to some degree (that is, $r_0 = 0$).

B consists of a non-null b_0, followed by a block of (M – N) '0's, followed by a block of nonzero coefficients, and terminated by a block of consecutive '0's. Blue has the same number of active strategies as does Red. Note that Blue may fail to attack the radar (that is, $b_0 > 0$).

Figures 2, 3, and 4 plot v and E_b for G(N, N + 1), G(N, N + 2), and G(N, N + 3) for small values of N; the staircase variation of E_b, associated with the number of active strategies, persists. The general trend may be guessed.

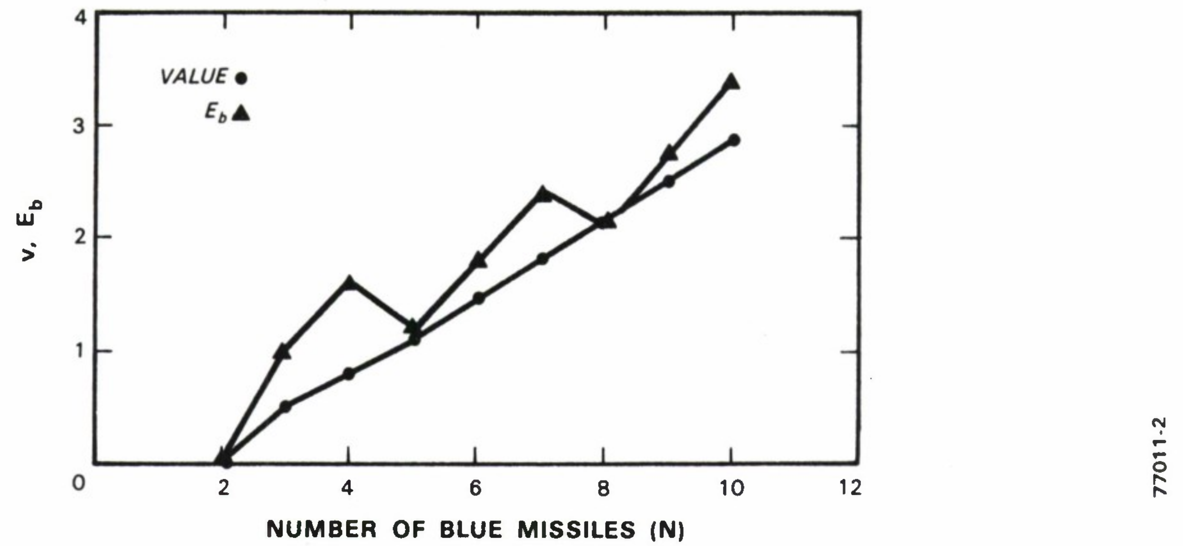

Figure 2. Value of game and number of radar missiles for game G(N, N + 1).

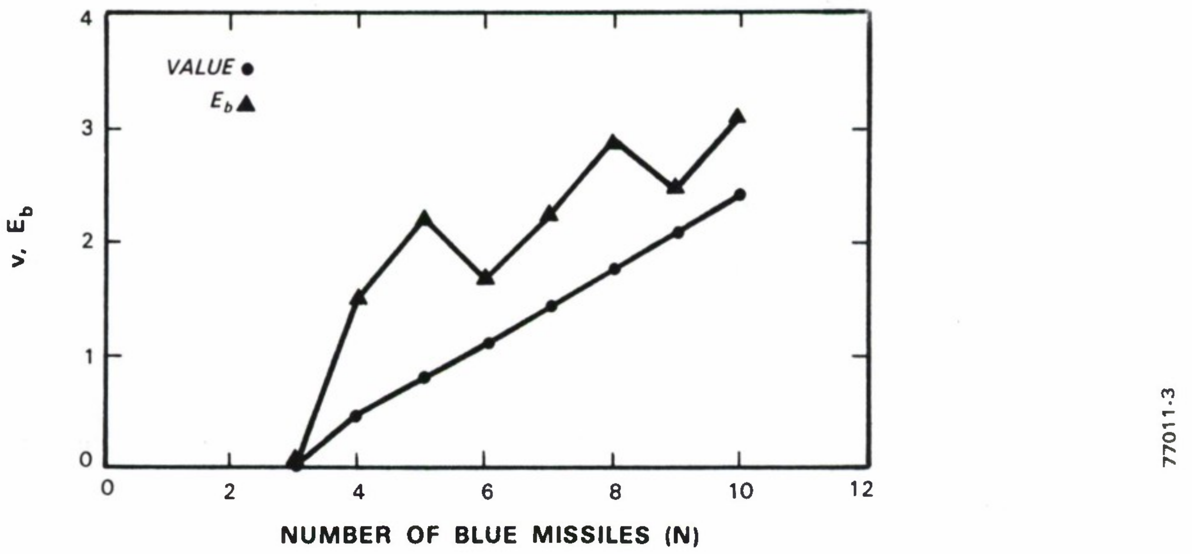

Figure 3. Value of game and number of radar missiles for game G(N, N + 2).

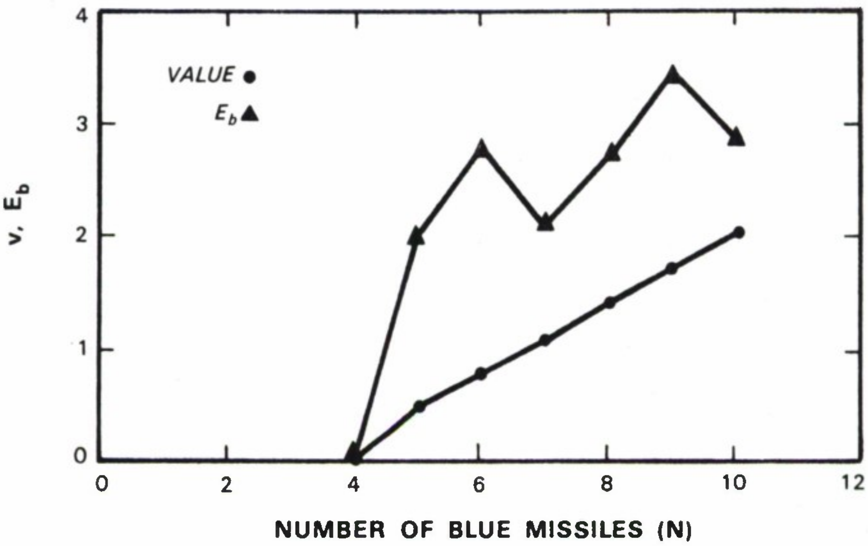

Figure 4. Value of game and number of radar missiles for game G(N, N + 3).

3.6.3. Large Values of (M - N)

For given N, the value of the game decreases monotonically with increasing M. If $M \geqslant 2N - 1$, we have $v = 0$, since, with that many weapons, Red can defend the radar with $N - 1$ interceptors while reserving N weapons to protect the base; in this way, Red suffers no loss whatever Blue does.

4. CONCLUDING REMARKS

We have defined Colonel Richard's Game and analyzed an elementary version of it. Further work will be directed at cases in which Blue has more than one boat with which to attack the radar and/or more than one boat with which to attack the base, and in which Red has more than one radar to defend its territory.

APPENDIX A: PROOF THAT $E_r = v$ FOR THE GAME G(N)

Consider the $(N + 1) \times (N + 1)$ matrix:

$$Q = \begin{bmatrix} 0 & 1 & 2 & 3 & \dots & N \\ .. & .. & .. & .. & & .. \\ .. & .. & .. & .. & & .. \end{bmatrix}$$

where the entries in the first row are as shown and the entries elsewhere may have any value, except that we assume that Q is nonsingular. We will show that Q, and therefore *a fortiori* G′, has what we will call the Unit Property:

$$KQ^{-1}J^t = 1 \quad .$$

Write the inverse of Q as:

$$Q^{-1} = [q_0 \quad q_1 \quad \dots \quad q_N]$$

where q_i is an $(N + 1)$-component column vector. Then, since:

$$K = [0 \quad 1 \quad 2 \quad \dots \quad N]$$

is the same as the first row of Q, we have, by definition of the inverse of a matrix:

$$Kq_0 = 1$$

$$Kq_1 = 0$$

$$\dots$$

$$Kq_N = 0$$

so that:

$$KQ^{-1} = [1 \quad 0 \quad \dots \quad 0]$$

and we have immediately, $KQ^{-1}J^t = 1$, the Unit Property.

This property arises from the fortuitous identity of the first line of the payoff matrix with the vector, K, used in calculating the expected number of interceptors defending the radar. From the fact that the payoff matrices, full or reduced, of Colonel Richard's Game G(N) have the Unit Property, we find (assuming that G′ has an inverse, which is always true for $N > 1$):

$$\begin{aligned} E_r &= RK^t = vJ(G^{-1})^t K^t \\ &= v(KG^{-1}J^t)^t \\ &= v \quad . \end{aligned}$$

Therefore, the expected number of interceptors that Red will use to protect the radar equals the value of the game.

APPPENDIX B: STRATEGY VECTORS FOR G(N)

This Appendix lists the values of B and R for $1 \leqslant N \leqslant 20$. 'Ac' indicates the number of active strategies; the number of inactive strategies is, of course, (N + 1 – Ac). The symbol, '. .]' indicates that the remainder of the vector is to be filled in with '0's. A positive entry that rounds off to 0.00 is indicated by '.0+'.

N	Ac	
1	1	B = [any]
		R = [0 1]
2	2	B = [.50 .50 . .]
		R = [.50 .50 . .]
3	3	B = [.33 .17 50 . .]
		R = [.33 .50 .17 . .]
4	3	B = [.50 .17 33 . .]
		R = [.25 .33 .42 . .]
5	4	B = [.40 .10 .17 .33 . .]
		R = [.20 .25 .33 .22 . .]
6	5	B = [.33 .07 .10 .17 .33 . .]
		R = [.17 .20 .25 .33 .05 . .]
7	5	B = [.43 .07 .10 .15 .25 . .]
		R = [.14 .17 .20 .25 .24 . .]
8	6	B = [.38 .05 .07 .10 .15 .25 . .]
		R = [.12 .14 .17 .20 .25 .12 . .]
9	7	B = [.33 .04 .05 .07 .10 .15 .25 . .]
		R = [.11 .12 .14 .17 .20 .25 .0+ . .]
10	7	B = [.40 .04 .06 .07 .10 .13 .20 . .]
		R = [.10 .11 .12 .14 .17 .20 .15 . .]
11	8	B = [.36 .04 .04 .06 .07 .10 .13 .20 . .]
		R = [.09 .10 .11 .12 .14 .17 .20 .06 . .]
12	8	B = [.42 .04 .04 .06 .07 .09 .12 .17 . .]
		R = [.08 .09 .10 .11 .12 .14 .17 .18 . .]
13	9	B = [.38 .03 .04 .04 .06 .07 .09 .12 .17 . .]
		R = [.08 .08 .09 .10 .11 .12 .14 .17 .10 . .]
14	10	B = [.36 .03 .03 .04 .04 .06 .07 .09 .12 .17 . .]
		R = [.07 .08 .08 .09 .10 .11 .12 .14 .17 .03 . .]

15	10	B = [.40 .03 .03 .04 .04 .06 .07 .08 .11 .14 . .] R = [.07 .07 .08 .08 .09 .10 .11 .12 .14 .13 . .]
16	11	B = [.38 .02 .03 .03 .04 .04 .06 .07 .08 .11 .14 . .] R = [.06 .07 .07 .08 .08 .09 .10 .11 .12 .14 .07 . .]
17	12	B = [.35 .02 .02 .03 .03 .04 .04 .06 .07 .08 .11 .14 . .] R = [.06 .06 .07 .07 .08 .08 .09 .10 .11 .12 .14 .01 . .]
18	12	B = [.39 .02 .03 .03 .03 .04 .04 .05 .06 .08 .10 .12 . .] R = [.06 .06 .06 .07 .07 .08 .08 .09 .10 .11 .12 .10 . .]
19	13	B = [.37 .02 .02 .03 .03 .03 .04 .04 .05 .06 .08 .10 .12 . .] R = [.05 .06 .06 .06 .07 .07 .08 .08 .09 .10 .11 .12 .04 . .]
20	13	B = [.40 .02 .02 .03 .03 .03 .04 .04 .05 .06 .07 .09 .11 . .] R = [.05 .05 .06 .06 .06 .07 .07 .08 .08 .09 .10 .11 .12 . .]

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GLOSSARY

Ac	Number of active strategies in game
Active Strategy	Strategy played with finite probability
B	Blue strategy vector. Components are b_i
BAB	Base-attack boat
Blue	Attacking force
E_b	Expected number of Blue missiles that attack radar
E_r	Expected number of Red missiles that defend radar
G	Payoff, or game, matrix
G′	Reduced matrix: game matrix G, with inactive strategies deleted
G(N, M)	Game matrix with N Blue and M Red missiles
G(N)	Game matrix with N Blue and N Red missiles [= G(N, N)]
g_{ij}	Gain to Blue if radar is attacked with i missiles and Red defends with j missiles
i	Number of Blue missiles that attack radar
Inactive Strategy	Strategy played with 0 probability
j	Number of Red missiles that defend radar
J	(N + 1)-component vector. Components: $j_i = 1$
K	(N + 1)-component vector. Components: $k_i = i$
M	Number of Red missiles
N	Number of Blue missiles
Q	Working matrix. Components are q_i (Appendix A)
R	Red strategy vector. Components are r_j
RAB	Radar-attack boat
Red	Defending force
t	Transpose of a matrix
v	Optimal value of an engagement

UNCLASSIFIED

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game theory	mixed strategy	offense-defense
two-person game	active strategy	interceptors
zero-sum game	inactive strategy	missiles
payoff matrix		

20. ABSTRACT (*Continue on reverse side if necessary and identify by block number*)

Colonel Richard's Game, a two-player, single-stage, zero-sum abstract game, is defined. It is based on a military offense-defense situation and is a variant of the classical Colonel Blotto's Game.

Colonel Richard's Game is described, an elementary version is solved, and extension to more complex versions discussed.